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On the retarded potentials of inhomogeneous ellipsoids and sources of arbitrary shapes in the three-dimensional infinite space

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Abstract

We analyze the infinite space solutions of the three-dimensional inhomogeneous wave equation (the 'retarded potentials' or 'causal propagators') for ellipsoidal sources and for sources of arbitrary shapes. The 'short-time characteristics' of the retarded potential for a spatially inhomogeneous source density of δ-shaped time profile is considered. It is found that, the short-time characteristics is governed by the spatial inhomogeneity of the source density in the immediate vicinity of a spacepoint.

Surface integral representations are derived for spatial inhomogeneous source regions of *ellipsoidal symmetry*. For spherical sources these integral representations yield closed form solutions for the retarded potentials. We find that the wave field inside a spherical source consists of an incoming and outgoing spherical wave package, whereas the external wave field consists of an outgoing spherical wave package only. Characteristic runtime and superposition effects are discussed. Moreover, a numerical technique based on Gauss quadrature is applied to generate the wave field for a cubic source. The integral representations derived for the retarded potentials of inhomogeneous ellipsoidal sources are consistent with results previously derived by the authors for the Helmholtz potentials of homogeneous ellipsoids and ellipsoidal shells [Michelitsch, T.M., Gao, H., Levin, V.M., 2003. On the dynamic potentials of ellipsoidal shells. Q. J. Mech. Appl. Math. 56 (4), 629]. The derived solutions are crucial for many problems of wave propagation and diffraction theory as they may occur in materials science. As an example we give a formulation for the solution of

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the *retarded Eshelby inclusion problem* due to spatially and temporally varying eigenfields in the elastic isotropic infinite medium.

Keywords: Wave equation; Retarded potential; Causal propagator; Ellipsoid; Wave propagation; Dynamic Green's function; Retarded Eshelby tensor

1. Introduction

The study of wave propagation effects is a fundamental issue for many applications in science and technology. Most of these problems can be reduced to the determination of the wave field in the surrounding space of an emitter of a given spatial geometry and density. Despite of the importance of the problem, there seems to be a lack of rigorous analytical approaches for the determination of retarded potentials in the literature. The goal of this paper is to develop analytical expressions for the retarded potentials due to a given spatial source distribution which may also be time dependent. Among all time dependent source distributions, those of δ-type time dependence represent an important class of problems which are worthy to be considered in detail. The goal of this paper is to derive compact integral representations for the retarded potential of this problem class when the source region has ellipsoidal symmetry. In electrodynamics this retarded potential plays a key role in the causal solution of the Maxwell equations describing electromagnetic wave (light) propagation and radiation, scattering and diffraction effects (see e.g. Jackson (1999)).

In materials science a vast literature is devoted to *static* problems. Examples can be found in the mechanics of materials (Eshelby tensor), in electrodynamics (electrostatic potentials due to a charge distribution), in astronomy (the gravitational potential caused by a mass distribution) and this list can be continued. The determination of *Newtonian potentials* has been considered as a key issue of theoretical physics in the 19th century. Key contributions to the subject were presented by several authors (Green, 1828; Caley, 1875a; Caley, 1875b; Ferrers, 1877; Dyson, 1891; Hobson, 1896; Levin and Muratov, 1971; and Rahman, 2001 among others referenced therein). Ferrers (1877) derived a method to obtain the potential for polynomial densities of ellipsoids and Dyson (1891) solved this problem in full generality in terms of a series of 1D-integrals. Dyson's solution was used by Levin and Muratov (1971) and Rahman (2001) to obtain the Newtonian potential of heterogeneous ellipsoids and ellipsoidal shells. Hobson (1896) treated the potential in the *n*-dimensional space. Eshelby (1957) showed that inclusion and inhomogeneity problems can be reduced to determine two types of scalar (static) potentials of the region covered by the inclusion or inhomogeneity. Based on Eshelby's classical work of 1957, much work was done to describe the mechanics of microinhomogeneous materials in the static framework. Based on Dyson's work (1891), Rahman (2002) developed an elegant representation for the elastic fields of an ellipsoidal inclusion with *polynomial eigenstrain*. There the elastic fields are expressed in terms of elliptic integrals. With this key solution the inclusion problem was solved in principle for an *inhomogeneous* ellipsoidal inclusion of *arbitrary density*.

In material science the Helmholtz potentials are key quantities for the solution of the dynamic variant of Eshelby's inclusion problem and the *dynamic* Eshelby tensor (Michelitsch et al., 2003a; Wang et al., 2003, in press). Despite the basic importance of the Helmholtz potential for dynamical problems, so far only little attention is paid in the literature due to the considerable mathematical complexity of these problems. There are only a few cases of source region geometries where closed form solutions of the inhomogeneous Helmholtz equation were found, namely for homogeneous spheres (Mikata and Nemat-Nasser, 1990) and for homogeneous continuous cylindrical fibers (Michelitsch et al., 2002, 2003a). Fu and Mura (1982) have considered integrals for the determination of the dynamic potential of an ellipsoid of inhomogeneous densities. However their analysis remains confined to the quasi-static limiting case. Michelitsch et al. (2003a) have derived the dynamic Eshelby tensor of the three dimensional infinite space due to homogeneous eigenstrain of an ellipsoidal source region. In that paper the authors derived 2D integral representations for the Helmholtz potential. Wang et al. (2003) derived integral representations for elliptical continuous fibers in piezoelectric material which correspond to a two-dimensional variant of the dynamic inclusion problem. As for the Newtonian potentials in statics (Dyson, 1891; Ferrers, 1877), the main goal is, from the esthetic and practical point of view, the determination of the dynamic potentials in terms of *one-dimensional* integrals. So far this goal is achieved only for the Helmholtz potential of a *homogeneous ellipsoidal shell* (Michelitsch et al., 2003b). As shown there, for vanishing frequency (static limit) the Helmholtz potential corresponds to the Newtonian potential of the problem.

Due to the increasing amount of applications of dynamic processes in inhomogeneous materials (e.g. "smart materials") in newly emerging technologies, e.g. "smart structures" and non-destructive evaluation (NDE), not only the frequency domain solution (Helmholtz potential) is of interest, its *causal time representation*, namely the *retarded potential*, is also of crucial importance. Therefore, the goal of this paper is to derive compact integral representations for retarded potentials of *inhomogeneous ellipsoidal source regions* and to derive closed form solutions in degenerate cases (spherical sources).

The paper is organized as follows: in Section 2 we derive a compact (two-dimensional) surface integral representation for the *retarded potential* of a spatially *inhomogeneous ellipsoidal* source region by evaluating the convolution of the retarded Green's function over the source density. This representation yields closed form results for the degenerate case when the source region becomes a (inhomogeneous) sphere. By means of this closed form expression, characteristic wave-propagation and superposition effects of the wave field are discussed.

The retarded potential of a source of arbitrary shapes is evaluated numerically in Section 3 by means of a source of cubic symmetry. The numerical scheme which is based on Gauss quadrature is useful to be applied to inhomogeneous sources of arbitrary shapes and source densities.

A general expression for the short-time characteristics of an arbitrary spatial source density of δ-type time dependence is derived in Section 4 and related to the local spatial inhomogeneity of the source density.

As an application in solid mechanics we give in Section 5 a formulation for the causal time-domain solution of the *dynamic variant of Eshelby inclusion problem* in terms of *retarded potentials*.

2. Retarded potential of an inhomogeneous ellipsoidal source distribution

The basic problem we consider is defined by an inhomogeneous wave equation in the infinite three-dimensional space and has the form

$$\left(\Delta - \frac{1}{c^2}\left[\frac{\partial}{\partial t} + \gamma\right]^2\right)\bar{g} + \rho(\mathbf{r}, t) = 0 \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplacian, $\mathbf{r} = (x_i) = (x, y, z)$ indicate spatial Cartesian coordinates and $\rho(\mathbf{r}, t)$ the source density; c denotes the constant wave propagation velocity and $\gamma > 0$ is a positive damping constant which guarantees causal behavior (see e.g. Michelitsch et al. (2002, 2003a)). We assume a source density $\rho(\mathbf{r}, t)$ with $\delta(t)$-time dependence, being non-zero only inside the ellipsoid having the form [1]

$$\rho(\mathbf{r}, t) = \Theta(1 - P)f(P^2)\delta(t) \tag{2}$$

where $\Theta(\xi)$ denotes the Heaviside step function [2] and

[1] The approach derived can easily be extended to general non-local cases $\rho = \rho(P^2)$ where the only constraint is $\int d^3\mathbf{r}\rho(\mathbf{r}) < \infty$.

[2] $\Theta(\xi) = 1$ if $\xi > 0$ and $\Theta(\xi) = 0$ if $\xi < 0$.

$$P^2 = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2} \tag{3}$$

$\mathbf{a} = (a_i)$ denotes the set of semi axes of the ellipsoid. $P < 1$ and $P > 1$ characterizes the internal and external space of the ellipsoid, respectively.

In view of (1) we rewrite $\bar{g} = e^{-\gamma t} g$ where g denotes the *retarded potential* and solves

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) g + \rho(\mathbf{r})\delta(t) = 0 \tag{4}$$

where we emphasize that we only look for the *causal* (retarded) solution of (4).

Generally the retarded potential g can be expressed as a convolution of the retarded Green's function $\hat{g}$ and the source density, namely

$$g(\mathbf{r}, \mathbf{a}, t) = \int\int \hat{g}(|\mathbf{r} - \mathbf{r}'|, t - t')\rho(\mathbf{r}', t')\mathrm{d}^3\mathbf{r}'\,\mathrm{d}t' \tag{5}$$

The retarded Green's function is given by (e.g. Jackson, 1999)

$$\hat{g}(r, t) = \frac{\delta\left(t - \frac{r}{c}\right)}{4\pi r} \tag{6}$$

and solves

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\hat{g}(r, t) + \delta(t)\delta^3(\mathbf{r}) = 0 \tag{7}$$

which describes a singular spherical wave being non-zero only at $t = \frac{r}{c}$ which is the runtime from the source point (located at the origin $r = 0$) to the sphere r on which the spacepoint $\mathbf{r}$ is located. A consequence of (6) is Huygen's principle which indicates that any source point emits *spherical* waves.

Singular waves of the form (6) are denoted in the following as 'elementary waves'. With the density (2),(5) takes the form (by putting $\mathbf{r}' = (x_i') = (P'a_i n_i')$, with $\mathbf{n}' \cdot \mathbf{n}' = 1$ and $\mathrm{d}^3\mathbf{r}' = a_1a_2a_3\mathrm{d}\Omega(\mathbf{n}')P'^2\mathrm{d}P'$) [3]

$$g(\mathbf{r}, \mathbf{a}, t) = \frac{a_1a_2a_3}{4\pi}\int_{|\mathbf{n}'|=1}\mathrm{d}\Omega(\mathbf{n}')\int_0^1 \frac{\delta(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r} - \mathbf{r}'|} f(P'^2)P'^2\,\mathrm{d}P' \tag{8}$$

To evaluate this integral we make use of the identity (we have to consider only the case $t > 0$ which is expressed by a prefactor $\Theta(t)$)

$$-\frac{\Theta(t)}{8\pi^2 c}\frac{\partial}{\partial t}\int_{|\hat{\mathbf{k}}|}\mathrm{d}\Omega(\hat{\mathbf{k}})\delta\left(t - \frac{\hat{\mathbf{k}}\cdot\boldsymbol{\xi}}{c}\right) = \frac{\Theta(t)}{4\pi\xi}\left(\delta\left(t - \frac{\xi}{c}\right) + \delta\left(t + \frac{\xi}{c}\right)\right) = \frac{\Theta(t)}{4\pi\xi}\delta\left(t - \frac{\xi}{c}\right) \tag{9}$$

where $\xi = |\boldsymbol{\xi}|$. With (9) we can write (8) in the form ($\mathbf{r}' = (P'a_i'n_i')$) by putting $\xi = |\mathbf{r} - \mathbf{r}'|$

$$g(\mathbf{r}, \mathbf{a}, t) = -\frac{\Theta(t)a_1a_2a_3}{8\pi^2 c}\int_0^1 f(P'^2)P'^2\,\mathrm{d}P'\frac{\partial}{\partial t}\int_{|\hat{\mathbf{k}}|}\mathrm{d}\Omega(\hat{\mathbf{k}})\int_{|\mathbf{n}'|=1}\mathrm{d}\Omega(\mathbf{n}')\delta\left(t - \frac{\hat{\mathbf{k}}\cdot(\mathbf{r} - \mathbf{r}')}{c}\right) \tag{10}$$

We evaluate first the second integral in this expression, namely (by putting $\mathbf{r}' \cdot \hat{\mathbf{k}} = s(\hat{\mathbf{k}})P'u$ with $s(\hat{\mathbf{k}}) = \sqrt{(a_1\hat{k}_1)^2 + (a_2\hat{k}_2)^2 + (a_3\hat{k}_3)^2}$ and $u = \cos\theta$, $\mathrm{d}\Omega(\mathbf{n}') = \mathrm{d}\theta\sin\theta\mathrm{d}\varphi$)

[3] Integrals $\int_{|\mathbf{m}|=1}\mathrm{d}\Omega(\mathbf{m})(\ldots)$ are performed on the surface of the unit sphere $\mathbf{m}\cdot\mathbf{m} = 1$.

$$\int_{|\mathbf{n}'|=1} \mathrm{d}\Omega(\mathbf{n}')\delta\left(t - \frac{\hat{\mathbf{k}}\cdot(\mathbf{r}-\mathbf{r}')}{c}\right) = 2\pi\int_{-1}^{1} \mathrm{d}u\delta\left(t - \frac{\hat{\mathbf{k}}\cdot\mathbf{r}}{c} + \frac{P's(\hat{\mathbf{k}})}{c}u\right)$$
$$= \frac{2\pi}{P's(\hat{\mathbf{k}})}\left\{\Theta\left(\frac{P's}{c} + t - \frac{\hat{\mathbf{k}}\cdot\mathbf{r}}{c}\right) - \Theta\left(-\frac{P's}{c} + t - \frac{\hat{\mathbf{k}}\cdot\mathbf{r}}{c}\right)\right\} \tag{11}$$

By using (11), expression (10) takes the form (where the symmetry of the integrand allows us to replace $\hat{\mathbf{k}} \to -\hat{\mathbf{k}}$)

$$g(\mathbf{r}, \mathbf{a}, t) = \int_0^1 f(P'^2)P'\Phi(\mathbf{r}, \mathbf{a}, t, P')\mathrm{d}P' \tag{12}$$

where

$$\Phi(\mathbf{r}, \mathbf{a}, t, P') = \frac{\Theta(t)a_1a_2a_3}{4\pi}\int_{|\hat{\mathbf{k}}|=1} \frac{\mathrm{d}\Omega(\hat{\mathbf{k}})}{s(\hat{\mathbf{k}})}\left(\delta\left(t - \frac{\hat{\mathbf{k}}\cdot\mathbf{r} + P's}{c}\right) - \delta\left(t + \frac{\hat{\mathbf{k}}\cdot\mathbf{r} + P's}{c}\right)\right) \tag{13}$$

We observe that $\Phi(\mathbf{r}, \mathbf{a}, t, P')P'$ corresponds to the retarded potential of an *ellipsoidal shell* of an ellipsoid with semi-axes $P'a_i$ solving the wave equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(\Phi(\mathbf{r}, \mathbf{a}, t, P')P') + \delta(P - P')\delta(t) = 0 \tag{14}$$

Relations (12) and (13) coincide with those obtained previously (see Michelitsch et al., 2003a,b) which were derived by means of Fourier transformation technique. By exploiting the relation

$$\int_0^1 f(P'^2)P'(\delta(P' - \xi) - \delta(P' + \xi)) = f(\xi^2)\xi\Theta(1 - \xi^2) \tag{15}$$

we obtain for (12) the two-dimensional integral

$$g(\mathbf{r}, \mathbf{a}, t) = \frac{ca_1a_2a_3\Theta(t)}{4\pi}\int_{|\hat{\mathbf{n}}'|=1} \frac{\mathrm{d}\Omega(\mathbf{n}')}{r'^3}(ct + \mathbf{n}'\cdot\mathbf{r})f\left(\frac{(ct + \mathbf{n}'\cdot\mathbf{r})^2}{r'^2}\right)\Theta\left(1 - \frac{(ct + \mathbf{n}'\cdot\mathbf{r})^2}{r'^2}\right) \tag{16}$$

where we have put $r' = \sqrt{a_1^2n_1'^2 + a_2^2n_2'^2 + a_3^2n_3'^2}$. Eq. (16) holds for the entire (internal and external) space. The advantage of expression (16) in contrast to the initial three-dimensional integration problem (8), is obvious. Unlike (8), Eq. (16) is only a two-dimensional integral with a non-singular integrand.

2.1. Spherical source

Let us now consider the degenerate case of a spherical source region with $a_i = a$ ($P = \frac{r}{a}$). In this case (16) can be evaluated in closed form and yields after a routine calculation

$$g(r, a, t) = \frac{ca^2\Theta(t)}{4r}\left\{\left[F(1) - F\left(\frac{(r - ct)^2}{a^2}\right)\right]\Theta\left(1 - \frac{(r - ct)^2}{a^2}\right)\right.$$
$$\left. - \left[F(1) - F\left(\frac{(r + ct)^2}{a^2}\right)\right]\Theta\left(1 - \frac{(r + ct)^2}{a^2}\right)\right\} \tag{17}$$

where F and f are related by $f(\lambda) = \frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda}$. Eq. (17) holds for both the internal and external space, and has the form $g = g^{(1)}(r + ct, a)/r + g^{(2)}(r - ct, a)/r$, which describes the superposition of an *outgoing* and an

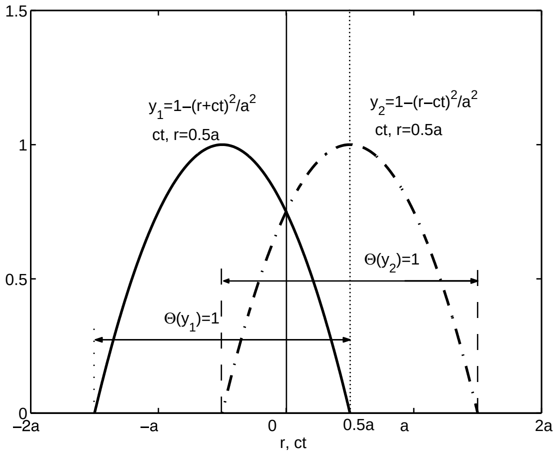

Fig. 1. Arguments y_i of the Θ-functions in Eq. (17). Spatial characteristics at $ct = 0.5a$ and time evolution at an *internal* spacepoint $r = 0.5a < a$, respectively.

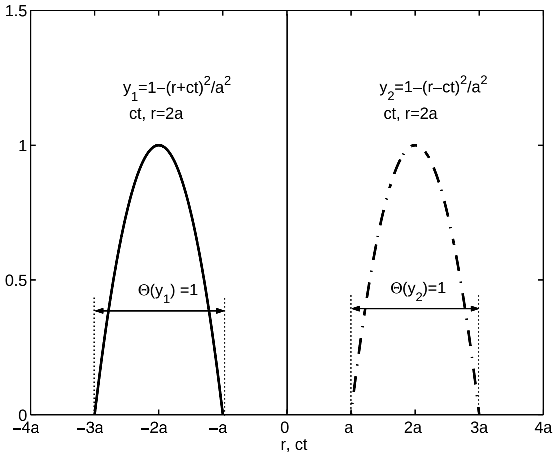

Fig. 2. Arguments y_i the Θ-functions in Eq. (17). Spatial characteristics for $ct = 2a$ and time evolution for an *external* spacepoint $r = 2a >$ ‘a, respectively, with $g_0^{\text{out}} = \frac{ca^2\Theta(t)}{4r} y_2$ (Eq. (24)).

incoming spherical wave. In Figs. 1 and 2 the positive parts of the parabolic arguments y_1 and y_2 of the Θ-functions of (17) are drawn. They propagate in the radial and anti-radial direction, respectively, indicating where the wave field is non-zero (Figs. 1,2). The quantities y_i are *symmetric* with respect to exchanging $r \leftrightarrow ct$. Hence Figs. 1 and 2 can be either conceived as y_i vs. ct-plots at a specific spacepoint (sphere) r or as y_i vs. r-plots at a specified ct. We observe that we have only in the internal space a non-zero overlap where incoming and outgoing waves interfere. This interference takes place only for times smaller than $2a/c$ which determines the time range of the existence of a non-vanishing wave field in the internal space.

For external spacepoints where no incoming wave $g^{(1)}/r$ can build up, the superposition of elementary waves (6) can cause an outgoing wave $g^{(2)}/r$ only. This is indicated in (17) by the vanishing of the second Θ-function for $r > a$.

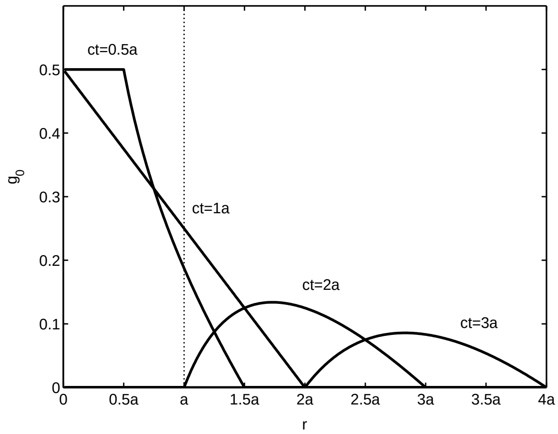

Fig. 3. Time evolution of the retarded potential g_0 of a homogeneous spherical source with radius a (Eq. (17) for $f(P^2) = 1$) in units of ca.

Hence we can summarize the following properties: in the internal space the wave field is governed by the superposition $g = g^{(1)}/r + g^{(2)}/r$ of an incoming and outgoing wave field as indicated by Fig. 1. [4] For runtimes greater than $2a/c$, i.e. when the wave field has left the source region, only the outgoing wave package exists, propagating with velocity c having wave-length $2a$ (Fig. 3). The vanishing of the incoming wave field can also be interpreted in view of Fig. 2: if $t > 2a/c$ the incoming wave field has moved into the (unphysical) region $r < 0$.

Let us confirm some necessary properties of (17) which follow directly from (8). To this end we consider the special case of a positive source function $f(\lambda) > 0$ for $0 < \lambda < 1$. According to relation (8), the retarded potential can only take positive values in this case. To confirm this we consider expression (17): for $f(\lambda) > 0$ its antiderivative $F(\lambda)$ is a monotonous increasing function with the property

$$\int_{\frac{(r-ct)^2}{a^2}}^{1} f(\lambda)\mathrm{d}\lambda = F(1) - F\left(\frac{(r-ct)^2}{a^2}\right) > 0, \quad \text{if} \quad 1 - \frac{(r-ct)^2}{a^2} > 0 \tag{18}$$

being a positive quantity if $\Theta\left(1 - \frac{(r-ct)^2}{a^2}\right) = 1$. For the external space $r > a$ we observe that the second Θ-function in (17) indicating the incoming wave, is identically vanishing $\forall t > 0$. Hence for the external space $r > a$ we have only

$$g^{\text{out}}(r,a,t) = \frac{ca^2\Theta(t)}{4r}\left[F(1) - F\left(\frac{(r-ct)^2}{a^2}\right)\right]\Theta\left(1 - \frac{(r-ct)^2}{a^2}\right) \tag{19}$$

which is according to (18) either positive or zero. Eq. (19) has the form $g^{\text{out}} = g^{(2)}(r - ct, a)/r$ which describes an outgoing spherical wave package propagating with velocity c without dispersion with stable wavelength $2a$. [5] The incoming wave $g^{(1)}(ct + r, a)/r$ of (17) is absent in the external space. As $\Theta\left(1 - \frac{(r-ct)^2}{a^2}\right) = 1$ only for $r - a < ct < a + r$ (Fig. 2), we observe from (19) that the outgoing wave arrives at the spacepoint $\mathbf{r}$ only at $t_1 = (r - a)/c$ which is the runtime from the closest source point on the boundary of the sphere. For $t < t_1$, due to the finite wave speed c, the wave has not yet reached the spacepoint $\mathbf{r}$ thus

[4] As ct and r occurs in the Θ-functions in (17) symmetrically we can interpret Figs. 1 and 2 as space or time plots.
[5] As c is a constant, no dispersion effects can occur.

$g^{\text{out}} = 0$. This runtime effect which is due to the outward propagation of the wave front also occurs in two dimensions (Wang et al., 2003).

In the *internal* space in the time range $0 < ct < a - r$ (see Fig. 1) both Θ-functions in (17) are equal to 1. Since $1 - \frac{(r-ct)^2}{a^2} > 1 - \frac{(r+ct)^2}{a^2}$, the outgoing and incoming waves then interfere and the sign of (17) is determined by

$$\begin{aligned} & F(1) - F\left(\frac{(r-ct)^2}{a^2}\right) - \left(F(1) - F\left(\frac{(r+ct)^2}{a^2}\right)\right) \\ & \quad = F\left(\frac{(r+ct)^2}{a^2}\right) - F\left(\frac{(r-ct)^2}{a^2}\right) \\ & \quad = \int_{\frac{(r-ct)^2}{a^2}}^{\frac{(r+ct)^2}{a^2}} f(\lambda)\mathrm{d}\lambda > 0 \end{aligned} \tag{20}$$

This integral is positive since $\frac{(r+ct)^2}{a^2} > \frac{(r-ct)^2}{a^2}$, i.e. also in the time range $0 < ct < a - r$.

For $a - r < ct < a + r$ is only $1 - \frac{(r-ct)^2}{a^2} > 0$, whereas $1 - \frac{(r+ct)^2}{a^2} < 0$ (see Fig. 1). Then the internal wave field consists of the outgoing wave only and is given by (19) which is positive because of (18) whereas the incoming wave is vanishing.

For $ct > a + r$ both Θ-functions are zero thus the wave field is identically vanishing. This is true for the internal and external space (see Figs. 1 and 2). This can be interpreted again as a runtime effect: for $ct > a + r$ which is larger than the runtime to the most distant source point, all elementary waves $\hat{g}(|\mathbf{r} - \mathbf{r}'|, t)$ emitted in the source region have passed by the spacepoint $\mathbf{r}$ and hence the wave field is vanishing. This effect is a speciality of the three-dimensional space and does not occur in two dimensions. [6] In two dimensions the wave field decays smoothly to zero for $t \to \infty$ instead of vanishing sharply at $ct = a + r$ (Wang et al., 2003). With (18) and (20) we have confirmed that the necessary condition $g(r, a, t) \geq 0$ if $f(P^2) > 0$ is fulfilled by expression (17) in the entire space. In conclusion of the above considerations we can rewrite the internal potential in the form

$$\begin{aligned} g^{\text{in}}(r, a, t) = \frac{ca^2\Theta(t)}{4r}\Bigg[& \left\{F\left(\frac{(r+ct)^2}{a^2}\right) - F\left(\frac{(r-ct)^2}{a^2}\right)\right\}\Theta(a - r - ct) \\ & + \left\{F(1) - F\left(\frac{(r-ct)^2}{a^2}\right)\right\}(\Theta(ct + r - a) - \Theta(ct - r - a)\Bigg] \end{aligned} \tag{21}$$

Especially interesting is the case of a homogeneous source $f(P^2) = 1$. Then (17) yields [7]

$$g_0(r, a, t) = \Theta(a - r)g_0^{\text{in}}(r, a, t) + \Theta(r - a)g_0^{\text{out}}(r, a, t) \tag{22}$$

$$g_0^{\text{in}}(r, a, t) = c^2 t\Theta(a - r - ct) + \frac{c}{4r}(a^2 - (ct - r)^2)(\Theta(ct + r - a) - \Theta(ct - r - a)) \tag{23}$$

If we consider a fixed spacepoint $r < a$ we see that (23) describes at first a linear increase in time for $0 < ct < a - r$. Then the wave field passes continuously into an outgoing wave package for $a - r < ct < a + r$ (Figs. 3 and 4). The solution for the *external space* then is given by (Eq. (19))

[6] Since the retarded 2D Green's function is $\hat{g}_2(|\mathbf{r} - \mathbf{r}'|, t) = \frac{\Theta(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{2\pi\sqrt{t^2 - \frac{|\mathbf{r}-\mathbf{r}'|^2}{c^2}}}$ tending smoothly to zero, but being non-zero $\forall t > |\mathbf{r} - \mathbf{r}'|/c$. A detailed discussion can be found in Wang et al., 2003.

[7] In the following we denote g_m the retarded potential of the source density P^{2m}, $m = 0, 1, 2, \ldots$

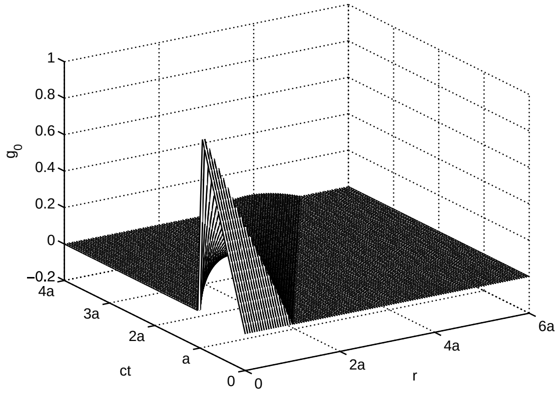

Fig. 4. Space-time plot of the retarded potential g_0 of a homogeneous spherical source ($f(P^2) = 1$, radius a) in units of ca.

$$g_0^{\mathrm{out}}(r,a,t) = \frac{ca^2\Theta(t)}{4r}\left[1 - \frac{(r-ct)^2}{a^2}\right]\Theta\left(1 - \frac{(r-ct)^2}{a^2}\right) \tag{24}$$

which is non-zero only in the time range $r - a < ct < r + a$ and describes an outgoing wave package of wave length $2a$.

It will be shown in Sections 3 and 4 that the *linear* characteristics in the time range $0 < ct < a - r$ is typical for a spatially *homogeneous* source (i.e. $f = 1$), independent on the shape of the inclusion. [8] Moreover, it was shown recently that this characteristics also occurs in two dimensions (Wang et al., 2003; Michelitsch et al., 2002). This linear characteristics can be physically interpreted as the homogeneous superposition of spherical elementary waves emitted on spheres of radii ct which are completely located *within* the source region. In Section 4 we will consider thoroughly the relation between the local spatial inhomogeneity of the source density and the short-time characteristics of the wave field immediately after the excitation at $t = 0$.

3. Retarded potentials of source regions with arbitrary shapes

In this section we utilize a numerical scheme which is useful to determine the retarded potentials of source regions with arbitrary shapes. This is demonstrated by means of a homogeneous source of cubic symmetry (Figs. 5 and 6). The numerical scheme consists in evaluating convolution (8). A detailed analysis of the wave field due to an arbitrary source will be given in Section 4.

The numerical scheme is useful for arbitrary spatial densities $\rho(\mathbf{r}, t)$ but we specialize it here to the cases $\rho(\mathbf{r}, t) = \rho(\mathbf{r})\delta(t)$.

First of all we consider the direct integration of the convolution (8), and specify the source region S as follows.

1. Its lower and upper limits in the x-direction, we denote as x_l, x_u.
2. Its lower and upper limits in the y-direction at a specified value of x, we denote as $y_l(x), y_u(x)$.
3. Its lower and upper limits in the z-direction at specified values of x and y, we denote as $z_l(x, y), z_u(x, y)$.

[8] Only the time range in which this behavior occurs depends on the shape and spacepoint.

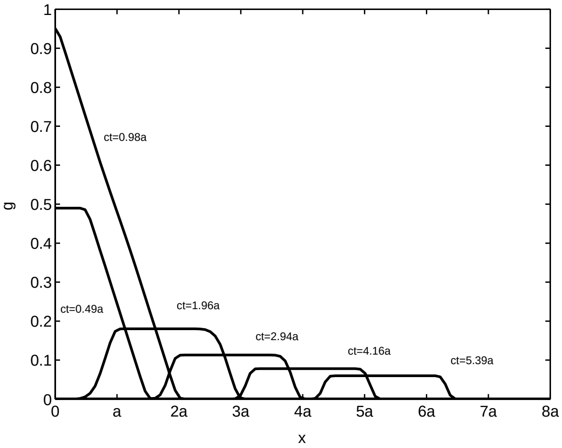

Fig. 5. Wave field of a homogeneous cubic source ($f(P^2) = 1$, $y = z = 0$). The cube is located at $|x_i| < a$ for different times in units of ca.

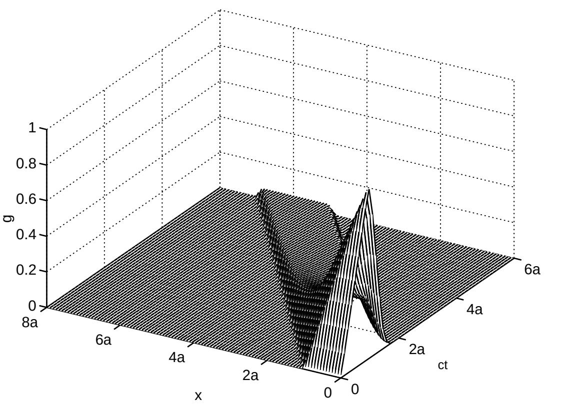

Fig. 6. Space-time plot of the wave field of a homogeneous cubic source ($f(P^2) = 1$, $y = z = 0$) in units of ca. The cube is located at $|x_i| < a$.

Once these lower and upper limits along different axes are known, according to (8), we have

$$g = \int_S \hat{g}(|\mathbf{r} - \mathbf{r}'|, t)\rho(\mathbf{r}')\mathrm{d}^3\mathbf{r}' = \int_{x_1}^{x_u} \mathrm{d}x' \int_{y_1(x')}^{y_u(x')} \mathrm{d}y' \int_{z_1(x',y')}^{z_u(x',y')} \mathrm{d}z' \hat{g}(|\mathbf{r} - \mathbf{r}'|, t)\rho(\mathbf{r}') \tag{25}$$

where we use the approximation

$$\hat{g}(r,t) = \frac{\delta(t - r/c)}{4\pi r} = \lim_{\varepsilon \to 0^+} \frac{\mathrm{e}^{-(t-r/c)^2/(4\varepsilon)}}{8\pi r\sqrt{\pi\varepsilon}} \tag{26}$$

To compute approximately the above integral, we use the Gauss–Chebyshev quadrature formula (e.g. Press et al., 1992), in which the abscissa and weights for the interval $[-1, 1]$ are

$$\begin{aligned} s_k &= \cos\left(\frac{\pi k - \pi/2}{N}\right), \quad k = 1, 2, \ldots, N \\ w_k &= \frac{\pi}{N} \end{aligned} \tag{27}$$

where N indicates the number of sampling points. Thus, for a function $h(x)$ defined within $[-1, 1]$, we have the integral formula of the form

$$\int_{-1}^{1} h(x)\,\mathrm{d}x \approx \sum_{k=1}^{N} w_k \sqrt{1 - s_k^2}\, h(s_k) \tag{28}$$

The quadrature is exact for all polynomials of degree $2N - 1$ (e.g. Press et al., 1992). In order to compute the integrals within a different domain, such as $[x_\mathrm{l}, x_\mathrm{u}]$, we introduce

$$\begin{aligned} s_k &= \frac{x_\mathrm{u} - x_\mathrm{l}}{2} \cos\left(\frac{\pi k - \pi/2}{N_x}\right) + \frac{x_\mathrm{u} + x_\mathrm{l}}{2}, \quad k = 1, 2, \ldots, N_x \\ w_k &= \pi \frac{x_\mathrm{u} - x_\mathrm{l}}{2N_x} \sqrt{1 - \cos^2\left(\frac{\pi k - \pi/2}{N_x}\right)} = \pi \frac{x_\mathrm{u} - x_\mathrm{l}}{2N_x} \sin\left(\frac{\pi k - \pi/2}{N_x}\right) \end{aligned} \tag{29}$$

and take into account that

$$\int_{x_\mathrm{l}}^{x_\mathrm{u}} h(x)\,\mathrm{d}x \approx \sum_{k=1}^{N_x} w_k h(s_k) \tag{30}$$

Furthermore, for domains $[y_\mathrm{l}(s_k), y_\mathrm{u}(s_k)]$ and $[z_\mathrm{l}(s_k, s_{kj}), z_\mathrm{u}(s_k, s_{kj})]$ we define

$$\begin{aligned} s_{kj} &= \frac{y_\mathrm{u}(s_k) - y_\mathrm{l}(s_k)}{2} \cos\left(\frac{\pi j - \pi/2}{N_y}\right) + \frac{y_\mathrm{u}(s_k) + y_\mathrm{l}(s_k)}{2} \\ w_{kj} &= \pi \frac{y_\mathrm{u}(s_k) - y_\mathrm{l}(s_k)}{2N_y} \sin\left(\frac{\pi j - \pi/2}{N_y}\right) \\ s_{kjm} &= \frac{z_\mathrm{u}(s_k, s_{kj}) - z_\mathrm{l}(s_k, s_{kj})}{2} \cos\left(\frac{\pi m - \pi/2}{N_z}\right) + \frac{z_\mathrm{l}(s_k, s_{kj}) + z_\mathrm{u}(s_k, s_{kj})}{2} \\ w_{kjm} &= \pi \frac{z_\mathrm{u}(s_k, s_{kj}) - z_\mathrm{l}(s_k, s_{kj})}{2N_z} \sin\left(\frac{\pi m - \pi/2}{N_z}\right) \end{aligned} \tag{31}$$

where $j = 1, 2, \ldots, N_y$ and $m = 1, 2, \ldots, N_z$. Then integral (25) becomes approximately ($\mathbf{r} \neq \mathbf{r}'_{kjm}$)

$$g(\mathbf{r}, t) \approx \sum_{k=1}^{N_x} \sum_{j=1}^{N_y} \sum_{m=1}^{N_z} w_k w_{kj} w_{kjm} \hat{g}(|\mathbf{r} - \mathbf{r}'_{kjm}|, t) \rho(\mathbf{r}'_{kjm}) \tag{32}$$

where $\mathbf{r}'_{kjm} = (s_k, s_{kj}, s_{kjm})$ and $\hat{g}(r, t)$ are defined by (26).

3.1. Cubic source

We demonstrate the efficiency of (32) by means of a homogeneous cubic source with density $\rho = \delta(t)\Theta(a^2 - x^2)\Theta(a^2 - y^2)\Theta(a^2 - z^2)$ which takes the value $\rho = \delta(t)$ inside the cube ($|x_i| < a$). In Fig. 5 the wave field is drawn for different times t after the excitation at $t = 0$ by employing (32) for $N_x = N_y = N_z = 120$. Fig 5 indicates for internal spacepoints at times $0 < ct < a - x$ a linear characteristics $c^2 t$ ($\gamma = 0$) which is a consequence that the source is spatially *homogeneous* inside the source. We will analyze the relation between the spatial inhomogeneity of the source and the short-time characteristics of the wave field in Section 4. Fig. 5 shows the propagation of the wave package for different times t along a symmetry-axis (x-axis) of the cube. Taking into account that the distance of the emission point $(a, 0, 0)$ of the cube in x-direction to its furthermost source points $(-a, \pm a, \pm a)$ is $\sqrt{6}a \approx 2.45a$, we can conclude that the wavelength of the projection of the wave package on the x-direction which occurs for $ct > ct_1 = \sqrt{6}a$ is given by this distance $\sqrt{6}a \approx 2.45a$. This is a consequence that any source point emits spherical singular elementary waves of the form (6) which reflects Huygen's principle. t_1 is the time which takes a spherical elementary wave from the source points $(-a, \pm a, \pm\, a)$ to the emission point $(a, 0, 0)$ along the x-axis. The wavelength $\sqrt{6}a \approx 2.45a$ of the projection of the wave field onto the x-axis is also indicated in Fig. 5 (see the wavelength of wave package at time $ct \approx 2.94a$). Fig. 6 shows the space–time plot of whole propagation process of this wave field.

This example indicates that relation (32) provides an efficient and accurate tool to approximate retarded potentials for sources of arbitrary shapes and densities.

4. Short-time characteristics for arbitrary spatial source densities

In this section we examine the retarded potential of a spatially arbitrary inhomogeneous source density distribution of the form $\rho(\mathbf{r}, t) = \delta(t)\rho(\mathbf{r})$ [9] in order to describe the characteristics of the wave field for "small" times t after the excitation at $t = 0$. The retarded potential of this problem is given by

$$g(\mathbf{r}, t) = \frac{c}{4\pi} \int \mathrm{d}^3\mathbf{r}' \frac{\delta(|\mathbf{r} - \mathbf{r}'| - ct)}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}') \tag{33}$$

It is convenient to introduce the following coordinates $\mathbf{r}' = \mathbf{r}_s + \mathbf{r}$ where $\mathbf{r}_s = r_s\hat{\mathbf{n}}$, with $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$ and $\mathrm{d}^3\mathbf{r}' = r_s^2 \mathrm{d}r_s \mathrm{d}\Omega(\hat{\mathbf{n}})$. Vector $\mathbf{r}_s$ parameterizes the space and we can write

$$g(\mathbf{r}, t) = \frac{c}{4\pi} \int_{|\hat{\mathbf{n}}|=1} \mathrm{d}\Omega(\hat{\mathbf{n}}) \int_0^\infty \rho(\mathbf{r} + r_s\hat{\mathbf{n}}) r_s \delta(ct - r_s) \mathrm{d}r_s \tag{34}$$

The r_s-integration yields the surface integral

$$g(\mathbf{r}, t) = \frac{c^2 t \Theta(t)}{4\pi} \int_{|\hat{\mathbf{n}}|=1} \mathrm{d}\Omega(\hat{\mathbf{n}}) \rho(\mathbf{r} + ct\hat{\mathbf{n}}) \tag{35}$$

The $\Theta(t)$-function indicates that the wave field exists only for $t > 0$ (causality). For a homogeneous density $\rho = 1$ (inside a certain source region S) the above linear characteristics $g_0 = c^2 t\Theta(t)$ is recovered by (35).

Expression (35) holds for an arbitrary spatial source distribution function $\rho(\mathbf{r})$. Especially for a localized source region, this integral holds for the internal and external space. Let us consider integral (35) in more detail. Assume that ρ is smooth enough that it can be expanded into a Taylor series [10]

$$\rho(\mathbf{r} + ct\hat{\mathbf{n}}) = \sum_{n=0}^{\infty} \frac{(ct)^n}{n!} (\hat{\mathbf{n}} \cdot \nabla_\mathbf{r})^n \rho(\mathbf{r}) \tag{36}$$

where $\nabla_\mathbf{r} = (\frac{\partial}{\partial x_i})$. We notice that only *even* powers $n = 2m$ of the series (36) [11] contribute to integral (35) so that we can write

[9] With the only constraint that the integral $\int \mathrm{d}^3\mathbf{r}\rho(\mathbf{r}) < \infty$.

[10] Generally this series converges only for enough "small" times $t > 0$, in this sense (36) describes only the "*short-time*" *characteristics* of the retarded potential.

[11] Corresponding to $\hat{\rho} = \frac{1}{2}(\rho(\mathbf{r} + ct\hat{\mathbf{m}}) + \rho(\mathbf{r} - ct\hat{\mathbf{m}}))$.

$$g(\mathbf{r},t) = \frac{c^2 t\Theta(t)}{4\pi}\sum_{m=0}^{\infty}\frac{(ct)^{2m}}{(2m)!}\int_{|\hat{\mathbf{n}}|=1} \mathrm{d}\Omega(\hat{\mathbf{n}})(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{r}})^{2m}\rho(\mathbf{r}) \tag{37}$$

Expanding $(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{r}})^{2m}$ into a multinomial series yields

$$g(\mathbf{r},t) = \frac{c^2 t\Theta(t)}{4\pi}\sum_{m=0}^{\infty}\frac{(ct)^{2m}}{(2m)!}\sum_{p'+q'+r'=2m}^{\infty}\frac{(2m)!}{p'!q'!r'!}I^{2m}_{p'q'r'}\frac{\partial^{2m}}{\partial x^{p'}\partial y^{q'}\partial z^{r'}}\rho(\mathbf{r}) \tag{38}$$

where

$$I^{2m}_{p'q'r'} = \int_{|\hat{\mathbf{n}}|=1}\mathrm{d}\Omega(\hat{\mathbf{n}})n_1^{p'}n_2^{q'}n_3^{r'} \tag{39}$$

and $p'+q'+r'=2m$. It can easily be shown when introducing spherical coordinates that $I^{2m}_{p'q'r'}$ is non-zero only if p',q',r' are all *even*, thus only terms with $p'=2p$, $q'=2q$, $r'=2r$ $(p+q+r=m)$ contribute (Michelitsch et al., 2003b). Hence (38) can be rewritten as

$$g(\mathbf{r},t) = \frac{c^2 t\Theta(t)}{4\pi}\sum_{m=0}^{\infty}\frac{(ct)^{2m}}{(2m)!}\sum_{p+q+r=m}^{\infty}\frac{2m!}{2p!2q!2r!}I^{2m}_{2p2q2r}\frac{\partial^{2m}}{\partial x^{2p}\partial y^{2q}\partial z^{2r}}\rho(\mathbf{r}) \tag{40}$$

Thus we only have to consider the integrals I^{2m}_{2p2q2r} $(p+q+r=m,\ m=0,1,2,\ldots)$ which we conveniently rewrite in the form

$$I^{2m}_{2p2q2r} = \int_{|\hat{\mathbf{n}}|=1}\mathrm{d}\Omega(\hat{\mathbf{n}})n_1^{2p}n_2^{2q}n_3^{2r} = \frac{1}{(2m)!}\frac{\partial^{2m}}{\partial\xi_1^{2p}\partial\xi_2^{2q}\partial\xi_3^{2r}}I_m \tag{41}$$

where we have introduced $\boldsymbol{\xi}=(\xi_i)$ and

$$I_m = \int_{|\hat{\mathbf{n}}|=1}\mathrm{d}\Omega(\hat{\mathbf{n}})(\boldsymbol{\xi}\cdot\hat{\mathbf{n}})^{2m} = \frac{4\pi}{(2m+1)}(\xi_1^2+\xi_2^2+\xi_3^2)^m \tag{42}$$

thus we find for (41) with $p+q+r=m$

$$I^{2m}_{2p2q2r} = \frac{4\pi}{(2m+1)}\frac{m!}{p!q!r!}\frac{(2p)!(2q)!(2r)!}{(2m)!} \tag{43}$$

Hence we can write for (40)

$$g(\mathbf{r},t) = c\Theta(t)\sum_{m=0}^{\infty}\frac{(ct)^{2m+1}}{(2m+1)!}\Delta^m\rho(\mathbf{r}) \tag{44}$$

where we have taken into account the multinomial expansion of the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ and $\Delta^0 =: 1$. It follows from (44) that the *short time-characteristics* of the wave field at a spacepoint $\mathbf{r}$ depends on the *local spatial characteristics* of the density function at this spacepoint. That is if the density has a local expansion (for $|\mathbf{r}-\mathbf{r}_0|\to 0$) of the form

$$\rho(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{(p'+q'+r')=n}a_n^{p'q'r'}(x-x_0)^{p'}(y-y_0)^{q'}(z-z_0)^{r'}, \tag{45}$$

then a non-vanishing *even* term $(x-x_0)^{2p}(y-y_0)^{2q}(z-z_0)^{2r}$ $(p+q+r=m,\ m=0,1,2,\ldots)$ of total order $2m$ contributes in the limiting case $|\mathbf{r}-\mathbf{r}_0|\to 0$ only with a power t^{2m+1} to the short-time characteristics of the retarded potential at spacepoint $\mathbf{r}$ (Eq. (44)). Moreover, we observe that terms in (45) of the form $(x-x_0)^{p'}(y-y_0)^{q'}(z-z_0)^{r'}$ where at least one of the numbers p',q',r' is *odd* do not contribute to the short-time characteristics (44) of the *potential* in the limiting case $|\mathbf{r}-\mathbf{r}_0|\to 0$. Furthermore, a monomial density $\rho(\mathbf{r}) = x^p y^q z^r$ $(p + q + r = n)$ yields a polynomial short-time characteristics at the spacepoint $\mathbf{r}$ of degree $2m + 1$ in time, where either $2m = n$ if n is even or $2m = n - 1$ if n is odd.

As an example we recall briefly Eq. (21) for a spherical source region with a density $\rho = \Theta(1 - P)P^{2m}$, $(P = r/a)$ and consider its short-time characteristics at $\mathbf{r} = 0$. The retarded potential (17) then yields

$$g_m^{\text{in}} = \begin{cases} \dfrac{c}{4(m+1)a^{2m}r}\left((ct+r)^{2m+2} - (ct-r)^{2m+2}\right), & 0 \leqslant ct \leqslant a - r \\ \dfrac{c}{4(m+1)a^{2m}r}\left(a^{2m+2} - (ct-r)^{2m+2}\right), & a - r \leqslant ct \leqslant a + r \\ 0, & a + r \leqslant ct \end{cases} \tag{46}$$

and for the external space $(r > a)$

$$g_m^{\text{out}} = \begin{cases} 0, & 0 \leqslant ct \leqslant r - a \\ \dfrac{c}{4(m+1)a^{2m}r}\left(a^{2m+2} - (ct-r)^{2m+2}\right), & r - a \leqslant ct \leqslant a + r \\ 0, & a + r \leqslant ct \end{cases} \tag{47}$$

The short time-characteristics is given by $(46)_1$ which holds for $(0 < ct < a - r)$. It follows that it is a polynomial of degree $2m + 1$ in time in the internal space. Let us consider the short-time characteristics in the origin $\mathbf{r} = 0$. In the vicinity of $\mathbf{r} = 0$ the expansion (45) of the density $\rho = (\frac{x_0^2}{a_1^2} + \frac{y_0^2}{a_2^2} + \frac{z_0^2}{a_3^2})^m$ contains only terms $(x - x_0)^{2p}(y - y_0)^{2q}(z - z_0)^{2r}$ $(\mathbf{r} = 0)$, thus according to our observation the short-term characteristics of the wave field at $\mathbf{r} = 0$ should be governed *only by the power* t^{2m+1}. In view of $(46)_1$ we indeed obtain for $r = 0$ the only term (for $0 < t < a/c$)

$$g_m(r = 0, t) = c\frac{(ct)^{2m+1}}{a^{2m}} \tag{48}$$

which is in full agreement with (44). The case $m = 0$ of a *homogeneous* sphere having a linear short-time characteristics $g_0(r = 0,t) = c^2 t$ (for $0 < t < a/c$) is also covered by (48).

5. Dynamic variant of Eshelby's inclusion problem

As an application of the retarded potentials derived in the above sections we consider here the retarded variant of the dynamic Eshelby's inclusion problem in the three-dimensional infinite linear elastic medium.

We consider a material with elastic constants C_{ijrs} and mass density ρ_{m} and assume that these material characteristics are identical in both the homogeneous matrix and the inclusion. This material system has the constitutive relations

$$\sigma_{ij} = C_{ijrs}(\epsilon_{rs} - \epsilon^*_{rs}) \tag{49}$$

where $\boldsymbol{\sigma}, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}^*, \boldsymbol{C}$ denote stress, strain, eigenstrain and the tensor of elastic constants, respectively, and

$$\epsilon_{rs} = \frac{1}{2}(\partial_r u_s + \partial_s u_r) \tag{50}$$

The inclusion S is assumed to undergo a *non-uniform* space–time transformation with an eigenstrain $\boldsymbol{\epsilon}^*$ of the form

$$\boldsymbol{\epsilon}^*(\mathbf{r}, t) = \rho(\mathbf{r}, t)\boldsymbol{\epsilon}^0 \tag{51}$$

with a "density" function ρ given by [12]

$$\rho(\mathbf{r},t) = \Theta_s(\mathbf{r})\chi(\mathbf{r},t) \tag{52}$$

where $\Theta_s(\mathbf{r})$ denotes the characteristic function of the inclusion [13] and ϵ^0 is a symmetric and *constant* tensor. In (52) we have introduced the *scalar* function $\chi(\mathbf{r},t)$ which characterizes the space–time variation of the eigenstrain. We assume the absence of external body forces. Then the equations of motion are given by

$$\rho_m \frac{\partial^2}{\partial t^2} u_i = \partial_j \sigma_{ij} \tag{53}$$

where ρ_m denotes the *mass density* of the material and $\mathbf{u}$ the displacement field. With (49) and (50) this equation assumes the form

$$\left(T(\nabla) - \rho_m \frac{\partial^2}{\partial t^2}\mathbf{1}\right)\mathbf{u}(\mathbf{r},t) + \mathbf{f}^*(\mathbf{r},t) = 0 \tag{54}$$

where $T_{ij}(\nabla) = C_{ikjl}\partial_k\partial_l$, $\mathbf{1}$ is the 3×3 unity tensor and $\boldsymbol{f}^*(\mathbf{r},t)$ is the effective force density that induces the same displacement field $\mathbf{u}$ in the undisturbed matrix as the inclusion and is given by

$$f_i^*(\mathbf{r},t) = -C_{ijrs}\epsilon^0_{rs}\partial_j\rho(\mathbf{r},t) \tag{55}$$

The physical displacement field $\mathbf{u}(\mathbf{r},t)$ can be expressed by the *retarded* Green's function in the form

$$\mathbf{u}(\mathbf{r},t) = \int\int \hat{G}(\mathbf{r}-\mathbf{r}',t-t')\mathbf{f}^*(\mathbf{r}',t')\mathrm{d}^3\mathbf{r}'\,\mathrm{d}t' \tag{56}$$

The retarded Green's function $\hat{G}(\mathbf{r},t)$ then is defined as the causal solution of

$$\left(T(\nabla) - \rho_m \frac{\partial^2}{\partial t^2}\mathbf{1}\right)\hat{G}(\mathbf{r},t) + \mathbf{1}\delta(t)\delta^3(\mathbf{r}) = 0 \tag{57}$$

where $\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$ and $\delta(t)$ denote δ-functions. To determine the displacement field $\mathbf{u}(\mathbf{r},t)$ it is convenient to introduce a tensor function $\boldsymbol{G}(\mathbf{r},t)$ which is defined by

$$\left(T(\nabla) - \rho_m \frac{\partial^2}{\partial t^2}\mathbf{1}\right)G(\mathbf{r},t) + \mathbf{1}\rho(\mathbf{r},t) = 0 \tag{58}$$

Obviously $\boldsymbol{G}$ and $\hat{G}$ are then related by

$$G(\mathbf{r},\xi,t) = \int\int \hat{G}(\mathbf{r}-\mathbf{r}',t-t')\rho(\mathbf{r}',t')\mathrm{d}^3\mathbf{r}'\,\mathrm{d}t' = \int_S\int_{-\infty}^{t} \hat{G}(\mathbf{r}-\mathbf{r}',t-t')\chi(\mathbf{r}',t')\mathrm{d}^3\mathbf{r}'\,\mathrm{d}t' \tag{59}$$

In $(59)_2$ the causality of the retarded Green's function $\hat{G}(\mathbf{r},t) = \Theta(t)\hat{G}(\mathbf{r},t)$ has been used (i.e. $\hat{G}(\mathbf{r},t) \neq 0$ only for $t > 0$). $\boldsymbol{\xi}$ indicates a set of geometric characteristics of the inclusion (for instance the semi-axes a_i in the case of an ellipsoidal inclusion).

Let us now confine to the isotropic medium with Lame constants λ and μ. Then (59) assumes the form (Wang et al., in press) [14]

$$G(\mathbf{r},\xi,t) = \frac{1}{\mu}g_2(\mathbf{r},\xi,t)\mathbf{1} + \frac{1}{\rho_m}\nabla\otimes\nabla\{h_1(\mathbf{r},\xi,t) - h_2(\mathbf{r},\xi,t)\} \tag{60}$$

[12] The derivation also holds for 'non-local' densities ρ with the only constraint $\iint\rho(\mathbf{r},t)\mathrm{d}^3\mathbf{r}\ \mathrm{d}t < \infty$.
[13] $\Theta_s(\mathbf{r}) = 1, \mathbf{r} \in S$ and $\Theta_s(\mathbf{r}) = 0, \mathbf{r} \notin S$.
[14] An analogous expression also holds in two dimensions (Michelitsch et al., 2002).

where the functions $g_i(\mathbf{r},\xi,t)$ and $h_i(\mathbf{r},\xi,t)$ are determined by a *retarded potential* of the form defined in (1) and solve

$$\left(\Delta - \frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}\right)g_i(\mathbf{r},\xi,t) + \rho(\mathbf{r},t) = 0 \tag{61}$$

and

$$\frac{\partial^2}{\partial t^2}h_i(\mathbf{r},\xi,t) = g_i(\mathbf{r},\xi,t) \tag{62}$$

where c_i denote the sound velocities of an isotropic medium

$$c_1 = \sqrt{\frac{(\lambda+2\mu)}{\rho_m}}, \quad c_2 = \sqrt{\frac{\mu}{\rho_m}} \tag{63}$$

corresponding to one longitudinal and two transversal acoustic waves. Now we can conveniently express the '*retarded Eshelby tensor*' in terms of the scalar potentials (61) and (62). With Eqs. (55), (56), and (59) we can write for the displacement field

$$u_1(\mathbf{r},\xi,t) = -C_{kjrs}\epsilon^0_{rs}\partial_j G_{kl}(\mathbf{r},\xi,t) \tag{64}$$

The strain can then be written as

$$\epsilon_{il}(\mathbf{r},\xi,t) = -C_{kjrs}\epsilon^0_{rs}(P_{ijkl}(\mathbf{r},\xi,t))_{(il)} \tag{65}$$

where (li) indicates symmetrization with respect to the subscripts il. In (65) we have introduced the tensor

$$P_{ijkl}(\mathbf{r},\xi,t) = \partial_i\partial_j G_{kl}(\mathbf{r},\xi,t) \tag{66}$$

where $G_{kl}(\mathbf{r},\xi,t)$ are the components of (60) for an isotropic medium.

Using (65) we define the *retarded Eshelby tensor* $\boldsymbol{S}(\mathbf{r},\xi,t)$ analogously to statics by

$$\epsilon_{il}(\mathbf{r},\xi,t) = S_{ilrs}(\mathbf{r},\xi,t)\epsilon^0_{rs} \tag{67}$$

where the *retarded Eshelby tensor* is given by

$$S_{ilrs}(\mathbf{r},\xi,t) = -C_{kjrs}(P_{ijkl}(\mathbf{r},\xi,t))_{(il)} \tag{68}$$

which is a tensor function in space and time. (68) shows that the retarded Eshelby tensor due to an inclusion of arbitrary shape and inhomogeneous eigenstrain of the form (51) for an elastic isotropic medium is completely determined by scalar *retarded potentials* considered in this paper.

Eqs. (64)–(68) together with the potential function (59) hold generally in linear elastic anisotropic infinite media. However, in the case of elastic anisotropy (60) is not valid and hence the Eshelby tensor (68) then cannot be expressed in terms of scalar retarded potentials.

6. Conclusions

The solution of the inhomogeneous wave equation (retarded potentials) is analyzed for spatially inhomogeneous densities $\rho(\mathbf{r},t) = \rho(\mathbf{r})\delta(t)$. The retarded potential due to a $\delta(t)$-type time dependence of a spatial inhomogeneous source is considered in this paper. It represents a basic quantity in the theory of wave propagation and diffraction and is of fundamental interest in mathematical physics. This retarded potential represents the integral kernel (propagator) for a source density of the form $\rho(\mathbf{r},t) = \rho(\mathbf{r})\lambda(t)$. The frequency Fourier transform of this retarded potential represents the solution of the corresponding Helmholtz potential which is crucial for the time-harmonic problem (Michelitsch et al., 2003b). As an example of interest for the dynamical modelling in the mechanics of solids the dynamic variant of Eshelby's inclusion problem has been considered in Section 5. It has been shown there ((60) ff.) that for an isotropic medium the *retarded Eshelby tensor* due to an inhomogeneous space–time eigenstrain distribution in the inclusion of the form (51), is related to scalar *retarded* potentials as they were considered in Sections 2–4.

In materials sciences the results of this paper can be used for the modelling of the effective *dynamic* characteristics of the material such as the dynamic moduli, the dispersion curves (in the frequency domain) and the overall dynamic response of the microinhomogeneous material. Beside that, the present study is useful for the description of phenomena of wave propagation, scattering and diffraction in heterogeneous (random) media.

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References

Caley, A., 1875a. A memoir on prepotentials. Philos. Trans. S. Lond. 165, 675–774.
Caley, A., 1875b. On the Potential of the Ellipse and Circle. Proc. Lond. Math. S. 6, 38–55.
Dyson, F.D., 1891. The potentials of ellipsoids of variable densities. Q. J. Pure Appl. Math. XXV, 259–288.
Eshelby, J.D., 1957. The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc. R. Soc. Lond. A 241, 376–396.
Ferrers, N.M., 1877. On the potentials of ellipsoids, ellipsoidal shells, elliptic laminae and elliptic rings of variable densities. Q. J. Pure Appl. Math. 14, 1–22.
Fu, L.S., Mura, T., 1982. Volume integrals associated with the inhomogeneous Helmholtz equation. Wave Motion 4, 141–149.
Green, G., 1828. An essay on the determination of the exterior and interior attractions of ellipsoids of variable densities. In: Ferrers, N.M. (Ed.), Mathematical Papers of George Green. Chelsea, New York.
Hobson, E.W., 1896. On some general formulae for the potentials of ellipsoids, shells and discs. Proc. Lond. Math. Soc. 27, 416–519.
Jackson, J.D., 1999. Classical Electrodynamics, third ed. Wiley, New York.
Levin, M.L., Muratov, R.Z., 1971. On the internal potential of heterogeneous ellipsoids. Astrophys. J. 166, 441–445.
Michelitsch, T.M., Gao, H., Levin, V.M., 2003a. Dynamic Eshelby tensor and potentials for ellipsoidal inclusions. Proc. R. Soc. Lond. A 459, 863–890.
Michelitsch, T.M., Gao, H., Levin, V.M., 2003b. On the dynamic potentials of ellipsoidal shells. Q. J. Mech. Appl. Math. 56 (4), 629–648.
Michelitsch, T.M., Levin, V.M., Gao, H., 2002. Dynamic potentials and Green's functions of a quasi-plane piezoelectric medium with inclusion. Proc. R. Soc. Lond. A 458, 2393–2415.
Mikata, Y., Nemat-Nasser, S., 1990. Elastic field due to a dynamically transforming spherical inclusion. J. Appl. Mech. ASME 57 (4), 845–849.
Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P., 1992. Numerical Recipes in Fortran 77: the Art of Scientific Computing, second ed. Cambridge University Press.
Rahman, M., 2001. On the Newtonian potentials of heterogeneous ellipsoids and elliptical discs. Proc. R. Soc. Lond. A 457, 2227–2250.
Rahman, M., 2002. The isotropic ellipsoidal inclusion with a polynomial distribution of eigenstrain J. Appl. Mech. Trans. ASME 69 (5), 593–601.
Wang, J., Michelitsch, T.M., Gao, H., 2003. Dynamic fiber inclusions with elliptical and arbitrary cross-sections and related potentials in a quasi-plane piezoelectric medium. Int. J. Solids Structures 40, 6307–6333.
Wang, J., Michelitsch, T.M., Gao, H., Levin, V.M., 2004. On the solution of the dynamic Eshelby inclusion problem for inclusions of various shapes. Kachanov, M. (guest ed.), Int. J. Solids Structures: Micromechanics of Materials , in press (special issue).